Solutions, Problem Set 3

#2–5 Spring 2015, Dr. Chatfield

2) 4–6

$$S = \sum_{N=0}^{\infty} \sum_{(n_j)}^{*} x_1^{n_1} x_2^{n_2}$$

$$\longrightarrow \sum_{N=0}^{4} \sum_{n_1, n_2 = 0}^{2}{}^{*} x_1^{n_1} x_2^{n_2}$$

$$= \underbrace{(x_1^0 x_2^0)}_{N=0} + \underbrace{(x_1^1 x_2^0 + x_1^0 x_2^1)}_{N=1} + \underbrace{(x_1^2 x_2^0 + x_1^1 x_2^1 + x_1^0 x_2^2)}_{N=2}$$

$$+ \underbrace{(x_1^2 x_2 + x_1 x_2^2)}_{N=3} + \underbrace{(x_1^2 x_2^2)}_{N=4}$$

$$= 1 + x_1 + x_2 + x_1^2 + x_1 x_2 + x_2^2 + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_2^2$$

$$\prod_{k=1}^{2} (1 + x_k + x_k^2) = (1 + x_1 + x_1^2)(1 + x_2 + x_2^2)$$

$$= 1 + x_1 + x_2 + x_1^2 + x_1 x_2 + x_2^2 + x_1 x_2^2 + x_1^2 x_2 + x_1^2 x_2^2$$

$$= S \quad \text{Q.E.D.}$$

Note: $1 + x_k + x_k^2 = \sum_{n_k=0}^{2} x_k^{n_k}$, so we have shown above that

$$\underbrace{\sum_{N=0}^{\infty} \sum_{n_1, n_2 = 0}^{2}{}^{*} x_1^{n_1} x_2^{n_2}}_{\sum_{N=0}^{\infty} \sum_{n_1, n_2 = 0}^{2}{}^{*} \prod_{k=1}^{2} x_k^{n_k}} = \prod_{k=1}^{2} \sum_{n_k=0}^{2} x_k^{n_k}$$

and thus $\displaystyle \sum_{N=0}^{\infty} \sum_{n_1, n_2 = 0}^{2}{}^{*} \prod_{k=1}^{2} x_k^{n_k} = \prod_{k=1}^{2} \sum_{n_k=0}^{2} x_k^{n_k}$

3) 4–7 $pV = kT\ln\Xi = \pm kT \sum_j \ln(1 \pm \lambda^{-\varepsilon_j/kT})$

replace Σ with $\int$ and use for density of states

$$\omega(\varepsilon)\,d\varepsilon = 2\pi\left(\frac{2m}{h^2}\right)^{3/2} V\varepsilon^{1/2}\,d\varepsilon$$

$$pV = \pm 2\pi\left(\frac{2m}{h^2}\right)^{3/2} VkT \int_0^\infty \varepsilon^{1/2}\ln(1 \pm \lambda e^{-\varepsilon/kT})\,d\varepsilon$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots \quad -1<x<1$$
$$= -\sum_j \frac{(-1)^j}{j}x^j$$

There for $\ln(1-x)$, $0<x<1$:

$$\ln(1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 \dots$$
$$= -\sum_j \frac{1}{j}x^j = -\sum_j \frac{(+1)^j}{j}x^j$$

$$\frac{p}{kT} = \pm 2\pi\left(\frac{2m}{h^2}\right)^{3/2}\int_0^\infty \varepsilon^{1/2}\left(-\sum_j \frac{(\mp 1)^j}{j}(\lambda e^{-\varepsilon/kT})^j\right)d\varepsilon$$

$$= \mp 2\pi\left(\frac{2m}{h^2}\right)^{3/2}\sum_j \lambda^j \frac{(\mp 1)^j}{j}\int_{\varepsilon=0}^\infty \varepsilon^{1/2} e^{-\frac{j}{kT}\varepsilon}\,d\varepsilon$$

$$u = \varepsilon^{1/2} \qquad \frac{du}{d\varepsilon} = \frac{1}{2}\varepsilon^{-1/2} \qquad d\varepsilon = 2\varepsilon^{1/2}du$$

$$\frac{p}{kT} = (\dots)\left[2\int_{u=0}^\infty e^{-\frac{j}{kT}u^2}du\right] \qquad \text{Table: } \int_0^\infty e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$= (\dots)\left(\frac{\pi kT}{j}\right)^{1/2}$$

NOT QUITE WHAT TEXT HA

$$= \mp 2\pi\left(\frac{2m}{h^2}\right)^{3/2}\sum_j \lambda^j \frac{(\mp 1)^j}{j}\left(\frac{\pi kT}{j}\right)^{1/2}$$

$$p = \mp 2\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\sum_j \frac{(\mp 1)^j \lambda^j}{j^{3/2}} = \mp\frac{2}{\Lambda^3}\sum_j \frac{(\mp 1)^j \lambda^j}{j^{3/2}}$$

41 4-12 $E = 2N_+\varepsilon_0 - N\varepsilon_0$

$$Q = \sum_{\text{states}} e^{-\beta(N_+\varepsilon_0 - N_-\varepsilon_0)} = \sum_{\text{levels}} \frac{N!}{N_+!(N-N_+)!}\, e^{-\beta(2N_+ - N)\varepsilon_0}$$

$$= e^{+N\beta\varepsilon_0} \sum_{N_+=0}^{N} \frac{N!}{N_+!(N-N_+)!}\, e^{-2\beta N_+}$$

Compare with q^N:

$$q = e^{+\beta\varepsilon_0} + e^{-\beta\varepsilon_0}$$

$$q^N = (e^{+\beta\varepsilon_0} + e^{-\beta\varepsilon_0})^N$$

$\frac{2!}{1!1!}$

1

1 1 $\quad \frac{2!}{0!2!}$

1 2 1

in general, coefficient is $\frac{N!}{N_1!N_2!}$

$$= \sum_{N_+=0}^{N} \left(e^{+\beta\varepsilon_0}\right)^{N-N_+} \left(e^{-\beta\varepsilon_0}\right)^{N_+} \cdot \frac{N!}{N_+!(N-N_+)!}$$

$$= e^{+N\beta\varepsilon_0} \sum_{N_+=0}^{N} \frac{N!}{N_+!(N-N_+)!}\, e^{-\beta(2N_+ - N)\varepsilon_0}$$

$\boxed{\text{Thus } Q = q^N}$

Why $Q \neq \frac{q^N}{N!}$?

$\Rightarrow$ Because these particles are distinguishable.

5) 4–19 $\bar{\varepsilon} = \frac{1}{q}\sum \varepsilon_i e^{-\beta\varepsilon_i} \qquad q = \sum e^{-\beta\varepsilon_i}$

$$\overline{\varepsilon^2} = \frac{1}{q}\sum \varepsilon_i^2 e^{-\beta\varepsilon_i} = \frac{-1}{q}\frac{d}{d\beta} q\bar{\varepsilon} = -\bar{\varepsilon}\frac{d\ln q}{d\beta} - \frac{d\bar{\varepsilon}}{d\beta}$$

$$= \bar{\varepsilon}^2 - \frac{d\bar{\varepsilon}}{d\beta}$$

$$\underbrace{\overline{\varepsilon^2} - \bar{\varepsilon}^2}_{\sigma_\varepsilon^2} = -\frac{d\bar{\varepsilon}}{d\beta}$$ (This follows derivation of σ_E^2 in ch. 3)

Now, $\bar{E} = N\bar{\varepsilon}$, so

$$\sigma_\varepsilon^2 = -\frac{1}{N}\frac{d\bar{E}}{d\beta} = \frac{kT^2}{N}\frac{d\bar{E}}{dT} = \frac{kT^2}{N}C_v$$

$$\sigma_E^2 = kT^2 C_v \Rightarrow \sigma_\varepsilon^2 = \frac{\sigma_E^2}{N}$$

$$\frac{\sigma_\varepsilon^2}{\bar{\varepsilon}^2} = \frac{\sigma_E^2/N}{(\bar{E}/N)^2} = N\frac{\sigma_E^2}{\bar{E}^2}$$

$$\frac{\sigma_\varepsilon}{\bar{\varepsilon}} = N^{1/2}\frac{\sigma_E}{\bar{E}}$$

Thus the relative fluctuation in ε, the energy of a single molecule, is $O(10^{12})$ times larger than the fluctuation in $\bar{E}$, the average energy for $N \sim 10^{24}$ (typical macroscopic size).

Earlier we showed for an ideal gas: $\frac{\sigma_E}{\bar{E}} = \sqrt{\frac{2}{3}}N^{-1/2}$

$\Rightarrow \sigma_\varepsilon = \sqrt{\frac{2}{3}}\,\bar{\varepsilon}$, so the fluctuation in $\bar{\varepsilon}$ is the same magnitude as $\bar{\varepsilon}$