1-25 Following the argument on pp. 10-11, the energy of N non-interacting particles in a box (cubic) of length a is:

$$E = \frac{h^2}{8ma^2} \sum_{j=1}^{N} (n_{x_j}^2 + n_{y_j}^2) = \frac{h^2}{8ma^2} \sum_{j=1}^{2N} s_j^2$$

We seek the number of states $\Phi(E)$ with energy $\leq E$. This will be proportional to the volume of a $2N$-dimensional sphere of radius R where

$$R^2 = \sum_{j=1}^{2N} s_j^2 = \frac{8ma^2E}{h^2}$$

~~However, the volume of a sphere~~
The volume of a $2N$-dimensional sphere is (prob. 1-24):

$$V = \frac{\pi^{2N/2}}{\Gamma(\frac{2N}{2}+1)} R^{2N} = \frac{\pi^N}{\Gamma(N+1)} R^{2N}$$

However, we only consider that portion of the sphere for which all s_j are positive:

$$\Phi(E) = \left(\frac{1}{2}\right)^{2N} \frac{\pi^N}{\Gamma(N+1)} R^{2N} = \frac{\pi^N}{\Gamma(N+1)} \left(\frac{2ma^2E}{h^2}\right)^N$$

The number of states with energy between E and $E + \Delta E$ is

$$\Omega(E,\Delta E) = \Phi(E+\Delta E) - \Phi(E)$$

$$= \frac{\pi^N}{\Gamma(N+1)} \cdot \left(\frac{2ma^2}{h^2}\right)^N \underbrace{\left[-E^N + (E+\Delta E)^N\right]}$$

$$= -E^N + \Big(E^N + \frac{N!}{(N-1)!1!} E^{N-1} \Delta E + \underbrace{\frac{N!}{(N-2)!2!} E^{N-2} (\Delta E)^2 \ldots}_{\text{negligible for } \Delta E << E}$$

$$= N E^{N-1} \Delta E$$

$$\Omega(E,\Delta E) = \frac{\pi^N}{\Gamma(N+1)} \left(\frac{2ma^2}{h^2}\right)^N N E^{N-1} \Delta E$$

$$\lim_{\Delta E \to dE} \Omega(E,\Delta E) = \rho(E)\, dE = \underbrace{\frac{N\pi^N}{\Gamma(N+1)} \left(\frac{2ma^2}{h^2}\right)^N E^{N-1}}_{\rho(E)} dE$$

$$\rho(E) = \frac{N\pi^N}{\Gamma(N+1)} \left(\frac{2ma^2}{h^2}\right)^N E^{N-1}$$

1-29 $dA = -p\,dV - S\,d\cancel{T}$

$$\Rightarrow \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V \quad \text{Maxwell relation}$$

$dE = T\,dS - p\,dV$

$$\Rightarrow T = \left(\frac{\partial E}{\partial S}\right)_V \quad -p = \left(\frac{\partial E}{\partial V}\right)_S$$

$$\left(\frac{\partial E}{\partial V}\right)_T = \left(\frac{\partial E}{\partial V}\right)_S + \left(\frac{\partial E}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T \quad \text{change of variable}$$

$$\therefore \left(\frac{\partial E}{\partial V}\right)_T - T\left(\frac{\partial p}{\partial T}\right)_V = -p \quad \text{Q.E.D.}$$

1-46 For this problem, we will assume normalized distribution, since they are identified as probability densities, i.e.

$$\int f_1(x)\,dx = \int f_2(y)\,dy = \int f(x,y)\,dx\,dy = 1$$

① Show $\bar{w} = \bar{x} + \bar{y}$ where $w = x + y$, given that x and y are independent

$$\bar{w} = \iint w\, f(x,y)\,dx\,dy = \iint (x+y)\, f_1(x) f_2(y)\,dx\,dy$$

$$= \underbrace{\int x f_1(x)\,dx}_{\bar{x}} \underbrace{\int f_2(y)\,dy}_{1} + \underbrace{\int y f_2(y)\,dy}_{\bar{y}} \underbrace{\int f_1(x)\,dx}_{1}$$

$$\therefore\ \bar{w} = \bar{x} + \bar{y} \quad \text{Q.E.D.}$$

② Show $\overline{(w-\bar{w})^2} = \overline{(x-\bar{x})^2} + \overline{(y-\bar{y})^2}$

$$\overline{(w-\bar{w})^2} = \iint \left[(x+y) - (\bar{x}+\bar{y})\right]^2 f_1(x)\, f_2(y)\,dx\,dy$$

$$= \iint \left[(x^2 + 2xy + y^2) + (\bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2) - 2(x+y)(\bar{x}+\bar{y})\right] f_1(x) f_2(y)\,dx\,dy$$

$$= \overbrace{\int x^2 f_1(x)}^{\overline{x^2}} \overbrace{\int f_2(y)\,dy}^{1} + 2 \overbrace{\int x f_1(x)\,dx}^{\bar{x}} \overbrace{\int y f_2(y)\,dy}^{\bar{y}} + \overbrace{\int y^2 f_2(y)\,dy}^{\overline{y^2}} \overbrace{\int f_1(x)\,dx}^{1}$$

$$+ \bar{x}^2 \overbrace{\int f_1\,dx \int f_2\,dy}^{1} + 2\bar{x}\bar{y} \overbrace{\int f_1\,dx \int f_2\,dy}^{1} + \bar{y}^2 \overbrace{\int f_1\,dx\, f_2\,dy}^{1}$$

$$- 2\bar{x} \overbrace{\int x f_1\,dx}^{\bar{x}} \overbrace{\int f_2\,dy}^{1} - 2\bar{y} \overbrace{\int y f_2\,dy}^{\bar{y}} \overbrace{\int f_1\,dx}^{1} - 2\bar{x} \overbrace{\int y f_2\,dy}^{\bar{y}} \overbrace{\int f_1\,dx}^{1}$$

$$- 2\bar{y} \overbrace{\int x f_1\,dx}^{\bar{x}} \overbrace{\int f_2\,dy}^{1}$$

$$\overline{(w-\bar{w})^2} = \overline{x^2} + 2\bar{x}\bar{y} + \overline{y^2} + \bar{x}^2 + 2\bar{x}\bar{y} + \bar{y}^2$$
$$- 2\bar{x}^2 - 2\bar{y}^2 - 2\bar{x}\bar{y} - 2\bar{x}\bar{y}$$

$$= \underbrace{(\overline{x^2} - \bar{x}^2)}_{\overline{(x-\bar{x})^2}} + \underbrace{(\overline{y^2} - \bar{y}^2)}_{\overline{(y-\bar{y})^2}}$$

$$\therefore\ \overline{(w-\bar{w})^2} = \overline{(x-\bar{x})^2} + \overline{(y-\bar{y})^2} \quad \text{Q.E.D.}$$

1–57 We want to maximize $f(P_j) = -\sum_{j=1}^{N} P_j \ln P_j$

subject to the constraint: $g(P_j) = \sum_{j=1}^{N} P_j - 1 = 0$

$\Rightarrow \frac{\partial}{\partial P_j}\left\{\left(-\sum P_j \ln P_j\right) - \alpha\left(\sum P_j - 1\right)\right\} = 0$

$-\ln P_j - \frac{P_j}{P_j} - \alpha = 0$ for $j = 1$ to N

$\ln P_j = -\alpha - 1$

$\therefore P_j = e^{-\alpha - 1} = \text{constant}$ Q.E.D.

1–56 Use $e^x = 1 + x + \frac{1}{2}x^2 + \ldots$

$$C_V = 3Nk\left(\frac{\Theta_E}{T}\right)^2 \frac{e^{\Theta_E/T}}{(e^{\Theta_E/T}-1)^2}$$

$$= 3Nk\left(\frac{\Theta_E}{T}\right)^2 \frac{1 + \frac{\Theta_E}{T} + \frac{1}{2}\left(\frac{\Theta_E}{T}\right)^2 + \ldots}{\left(\frac{\Theta_E}{T} + \frac{1}{2}\left(\frac{\Theta_E}{T}\right)^2 + \ldots\right)^2}$$

In high-T limit, higher-order terms can be neglected;

$$C_V = 3Nk\left(\frac{\Theta_E}{T}\right)^2 \frac{1}{\left(\frac{\Theta_E}{T}\right)^2} = 3Nk$$

In low-T limit, $e^{\Theta_E/T} - 1 \rightarrow e^{\Theta_E/T}$ (can neglect "1"), and original expression becomes

$$C_V = \frac{3Nk\left(\frac{\Theta_E}{T}\right)^2}{e^{\Theta_E/T}} = 3Nk\left(\frac{\Theta_E}{T}\right)^2 e^{-\Theta_E/T}$$

For Debye model, $C_v = 9Nk\left(\frac{T}{\Theta_D}\right)^3 \int_0^{\Theta_D/T} \frac{x^4 e^x}{(e^x-1)^2}\,dx$

In ~~high~~-T limit, this becomes

$$C_v = 9Nk\left(\frac{T}{\Theta_D}\right)^3 \underbrace{\int_0^{\Theta_D/T} \frac{x^4(1+\dots)}{(x+\dots)^2}\,dx}_{\int x^2 dx \,=\, \frac{1}{3}x^3\Big|_0^{\Theta_D/T}}$$

$= 3Nk$ same as Einstein model

The low-T limit is less clear because the value of x in the integrand is not always either very large or very small. If though, we assume that the portion of the integral corresponding to small x can be neglected, we obtain:

$$\int_0^{\Theta_D/T} \frac{x^4 e^x\,dx}{(e^x-1)^2} = \underbrace{\int_0^{c} \frac{x^4 e^x\,dx}{(e^x-1)^2}}_{A} + \underbrace{\int_c^{\Theta_D/T} \frac{x^4 e^x\,dx}{(e^x-1)^2}}_{B}$$

If $A \ll B$

$$C_v = 9Nk\left(\frac{T}{\Theta_D}\right)^3 \underbrace{\int_c^{\Theta_D/T} \frac{x^4 e^x\,dx}{(e^x-1)^2}}_{\approx \int_c^{\Theta_D/T} x^4 e^{-x}\,dx} \approx 9Nk\left(\frac{T}{\Theta_D}\right)^3 \int_0^{\Theta_D/T} x^4 e^{-x}\,dx$$

2-5 $P_j = \dfrac{e^{-\beta E_j}}{\sum e^{-\beta E_j}}$ $\qquad \sum e^{-\beta E_j} = Q$

$$P_j \ln P_j = P_j \left[\ln (e^{-\beta E_j}) - \ln Q \right]$$

$$= -\beta E_j \frac{e^{-\beta E_j}}{\sum e^{-\beta E_j}} - \frac{e^{-\beta E_j}}{\sum e^{-\beta E_j}} \ln Q$$

$$\sum P_j \ln P_j = -\beta \underbrace{\frac{\sum E_j e^{-\beta E_j}}{\sum e^{-\beta E_j}}}_{\bar{E}} - \underbrace{\frac{\sum e^{-\beta E_j}}{\sum e^{-\beta E_j}}}_{1} \ln Q$$

$$\therefore -k \sum P_j \ln P_j = \frac{\bar{E}}{T} + k \ln Q = S \qquad \text{Q.E.D.}$$

2-6 $f(P_j) = \sum P_j \ln P_j$

constraints: $\sum P_j - 1 = 0$, $\sum E_j P_j - E = 0$

To maximize f, set

$$\frac{\partial}{\partial P_j}\left\{\sum P_j \ln P_j - \alpha\left(\sum P_j - 1\right) - \beta\left(\sum E_j P_j - E\right)\right\} = 0$$

$$\ln P_j + 1 - \alpha - \beta E_j = 0$$

$$\ln P_j = \alpha - 1 + \beta E_j$$

$$P_j = e^{\alpha - 1 + \beta E_j}$$

Note that P_j now depends on E_j (exponentially), whereas in problem 1-51, P_j were constant. The difference is due to the second constraint.

2-14 $\bar{E} = kT^2 \left(\frac{\partial \ln Q}{\partial T}\right)_{N,V}$

$$\ln Q = -\ln N! + \frac{3N}{2} \ln\left(\frac{2\pi mkT}{h^2}\right) + N \ln V$$

$$\left(\frac{\partial \ln Q}{\partial T}\right)_{N,V} = \frac{3N}{2} \cdot \frac{h^2}{2\pi mkT} \cdot \frac{2\pi mk}{h^2} = \frac{3N}{2T}$$

$\bar{E} = \frac{3}{2} NkT$ same as for ideal gas

$$\bar{p} = kT \left(\frac{\partial \ln Q}{\partial V}\right)_{N,T}$$

$$\left(\frac{\partial \ln Q}{\partial V}\right)_{N,T} = \frac{N}{V}$$

$\bar{p} = \frac{NkT}{V}$ equivalent to ideal gas law

Note if $Q = f(T) V^N$, $\ln Q = \ln f(T) + N \ln V$

$$\left(\frac{\partial \ln Q}{\partial V}\right)_{N,T} = \frac{N}{V}$$

Thus $\bar{p} = kT \left(\frac{\partial \ln Q}{\partial V}\right)_{N,T} = \frac{NkT}{V}$ which is ideal gas law

Best strategy: Never pick the 1st number. Thereafter, if a number is larger than all preceding numbers, pick it. Obviously pick the last number if you don't pick any other.

We can check this out by computing the probabilities of winning and losing with this strategy. By comparison, the "dumb" strategy of random choice has probability of winning $P_w = \frac{1}{4}$, probability of losing $P_\ell = \frac{3}{4}$

For "best" strategy, consider the probability that the strategy will cause you to pick a particular number and win or lose. Let $N1 = 1^{st}$ number, $N2 = 2^{nd}$ number etc.

$N=1$ Never pick, so $P_w = 0$ $P_\ell = 0$

$N=2$ Pick and win if $N2$ is largest $P_w = \frac{1}{4}$
Pick and lose if
$N2$ is 2^{nd} largest and $N1$ is 3^{rd} or 4^{th} largest $P = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$
$N2$ is 3^{rd} largest and $N1$ is 4^{th} largest. $P = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

$P_\ell = \frac{1}{4}$

$N=3$ Pick and win if $N3$ is largest of $\{N2, N3, N4\}$ and $N2 < N1$

$$P_W = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

Pick and lose if $N4$ is largest and $N3$ is 2nd largest and $N2 < N1$ $\quad P_\ell = \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$

$N=4$ Pick and win if $N4$ is largest and $N1$ is 2nd largest and $N3 < N2$ $\quad P_W = \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{24}$

Pick and lose if $N1$ is largest $\quad P_\ell = \frac{1}{4}$

	P_W	P_ℓ
$N=1$	0	0
2	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{6}$	$\frac{1}{24}$
4	$\frac{1}{24}$	$\frac{1}{4}$
	$\frac{11}{24}$	$\frac{13}{24}$

$$\frac{11}{24} + \frac{13}{24} = \frac{24}{24} = 1 \checkmark$$

The total probability of winning with the best strategy is $\frac{11}{24}$, considerably better than random.